Vizard Mathematics

.1 LEONHARD EULER

Vhile it is general practice today to date the beginning of the modern theory f complex numbers from the appearance of Wessel's paper, it is a fact that nany of the particular properties of $\sqrt{-1}$ were understood long before Vessel. The Swiss genius Leonhard Euler (1707–83), for example, knew of ne exponential connection to complex numbers. The son of a rural pastor, he riginally trained for the ministry at the University of Basel, receiving, at age eventeen, a graduate degree from the Faculty of Theology. Mathematics, owever, soon became his life's passion. He remained a pious man, but there vas never any doubt that he was, first, a mathematician.

Nothing could keep him from doing mathematics, not even blindness for ne last seventeen years of his life. Euler had a marvelous memory—it was aid he knew the *Aeneid* by heart—and so after losing his sight he simply did nonstrously difficult calculations in his head. His reputation among his con-emporaries was such that he was known as "analysis incarnate." Many years fter his death the nineteenth-century French astronomer Dominique Arago aid of him "Euler calculated without apparent effort, as men breathe, or as agles sustain themselves in the wind." When he died he had written more rilliant mathematics than had any other mathematician, and to this day he till holds that record.

While a student at Basel Euler studied with the mathematician John Ber-oulli (1667–1748) and, along the way, became friends with two of his sons, licolas and Daniel, who were also mathematicians. Several years older than uler, both soon recognized the younger man's talents, and so when the two ernoulli boys went off to the Imperial Russian Academy of Sciences in St. etersburg in 1725 they began to lobby for a spot there for Euler as well. licolas died in 1726, but Daniel continued his efforts and in 1727 Euler, too, rrived in Russia. This first of two stays in Russia would see his first great uccess, and in just a few years (1731) he was named an Academy Professor.

A few days before Euler first set foot in Russia, however, the Tsarina Cath-rine I (widow of Peter the Great) died and the throne passed to a twelve year ld boy. The regency that then ran the country had little sympathy for the ntellectual and expensive Academy, which was viewed as a collection of

foreign scientists meddling in Russian culture, and Euler no doubt found the place less than totally congenial. When Euler was invited by Frederick the Great of Prussia to leave the Russian Academy and to take up a similar post in the Berlin Academy, he was happy to accept, and there he stayed from 1741 to 1766. He left Berlin because four years earlier Catherine the Great ascended the Russian throne, the intellectual climate there once again became attractive (and Euler was allowed to write his own, generous contract), and his personal relationship with Frederick had deteriorated. And so Euler returned to St. Petersburg. There he remained until his sudden death of a stroke as he sat one evening doing what he loved most—mathematics.

6.2 EULER'S IDENTITY

In a letter dated October 18, 1740 to his one-time teacher John Bernoulli, Euler stated that the solution to the differential equation

$$\frac{d^2y}{dx^2}+y=0, \quad y(0)=2, \text{ and } \; y'(0)=0$$

(where the prime notation denotes differentiation) can be written in two ways; namely,

$$y(x)=2\cos(x),$$
$$y(x)=e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}.$$

The truth of Euler's statement is evident by direct substitution into the differential equation, and the evaluation of each $y(x)$ for the given $x = 0$ conditions. Euler therefore concluded that these two expressions, each apparently so unlike the other, are in fact equal, i.e., that

$$2\cos(x) = e^{ix} + e^{-ix}.$$

It is evident from that same letter that Euler also knew that

$$2i \sin(x) = e^{ix} - e^{-ix}.$$

Just a year after his letter to Bernoulli, Euler wrote another letter (dated December 9, 1741) to the German mathematician Christian Goldbach, in which he observed the near-equality

$$\frac{2^{\sqrt{-1}}+2^{-\sqrt{-1}}}{2}\approx\frac{10}{13}.$$

[illegible] easy to show that the left-hand side is cos(ln 2); that and $\frac{10}{13}$ do not begin to differ until the sixth decimal place—only a genius or a quack would notice such a thing, I think, and Euler was no quack!

One mathematician who definitely *did* have something of the quack to him, and who was fascinated by the mystical appearance of the mathematical symbols in Euler's equations, was the Polish-born Józef Maria Hoëné-Wroński (1776–1853), who became a French citizen. He once wrote that the number π is given by the astounding expression

$$\frac{4\infty}{\sqrt{-1}}\left\{(1+\sqrt{-1})^{1/\infty}-(1-\sqrt{-1})^{1/\infty}\right\}.$$

What could he have meant by writing such a thing? Wroński's entry in the *Dictionary of Scientific Biography* uses such words as "psychopathic" and "aberrant," and notes that he had "a troubled and deceived mind," but if one replaces all the infinity symbols with n, writes $(1 \pm i)$ in polar form, i.e., as $\sqrt{2}e^{\pm i\pi/4}$, uses Euler's formulas to expand the complex exponentials, and finally takes the limit as $n \to \infty$, then Wroński's bizarre expression does reduce to 2π. (Not π, as claimed, but perhaps Wroński was thinking of the leftmost infinity as being generated by $\frac{1}{2}n$, not just n—who can say now *what* that odd thinker was thinking?)[1]

Finally, in 1748, Euler published the explicit formula

$$e^{\pm ix} = \cos(x) \pm i\sin(x)$$

in his book *Introductio in Analysis Infinitorum*. To mathematicians, electrical engineers, and physicists this is universally known today as Euler's identity, but as you will soon see he was not the first to either derive it or publish it.

Euler's confidence in this astonishing expression was enhanced by his knowledge of the power series expansion of e^y,

$$e^y = 1+y+\frac{1}{2!}y^2+\frac{1}{3!}y^3+\frac{1}{4!}y^4+\frac{1}{5!}y^5+\cdots.$$

If you set $y = ix$, then

$$e^{ix} = 1+(ix)+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\frac{1}{4!}(ix)^4+\cdots.$$

I am admittedly using this power series expansion for e^y, with y real, in a rather daredevil manner when I substitute an imaginary quantity for y. I am ignoring the question of convergence. I'm doing this because the issue is addressed in great detail in any good book on analysis, where it is shown that

the series converg[illegible] complex values, and I simply don't want to turn this book into a textbook. Rest assured, all of the series that I write and treat as convergent in this book *do* converge.

Continuing by collecting real and imaginary parts, we arrive at

$$e^{ix} = \left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots\right)+i\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots\right).$$

But the expressions in the parentheses are the power series expansions for $\cos(x)$ and $\sin(x)$, respectively (known to mathematicians since at least Newton's time), and so Euler's identity is derived in a new way. The sine series, by the way, provides the proof to a statement I asked you just to accept back in section 3.2. Thus we have

$$\frac{\sin(x)}{x} = 1-\frac{1}{3!}x^2+\frac{1}{5!}x^4\cdots$$

and so, with $x = (1/2^n)\,\theta$, we have

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{2^n}\theta\right)}{\left(\frac{1}{2^n}\theta\right)} = 1,$$

as claimed.

The power series expansion of e^y was used by both Bernoulli and Euler in some breathtaking calculations. In 1697, for example, John Bernoulli used it to evaluate the mysterious-looking integral $\int_0^1 x^x dx$. Here's how he did it. First, using the trick I used in box 3.2 to calculate $(1+i)^{1+i}$, he wrote

$$x^x = e^{\ln(x^x)} = e^{x\ln(x)},$$

and then set $y = x\ln(x)$. The power series expansion then gave him

$$\int_0^1 x^x dx = \int_0^1\left\{\sum_{k=0}^{\infty}\frac{(x\ln x)^k}{k!}\right\}dx = \sum_{k=0}^{\infty}\frac{1}{k!}\left\{\int_0^1 (x\ln x)^k dx\right\}.$$

Using integration by parts it is easy to show that

$$\int_0^1 (x\ln x)^k dx = \frac{(-1)^k k!}{(k+1)^{k+1}},$$

a result which is not hard to arrive at if you remember or look up, at the proper time, $\lim_{x\to 0} x\ln x = 0$. From this it immediately follows that

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots = 0.78343\cdots.$$

6.3 Euler Makes His Name

Bernoulli's integral was a brilliant calculation, but his former student Euler far surpassed that achievement by using the power series expansion of sin(y), the imaginary part of e^{iy}, to accomplish what today is still considered a world-class *tour de force*. All he did was solve a problem that had stumped mathematicians for centuries! It also led him to write down a new function, called the *zeta function* today, that is behind the greatest unsolved problem in all of complex number theory; indeed, in all of mathematics. And that was so even before Fermat's last theorem was laid to rest in 1995. Here's what he did.

A mathematical problem of long standing has been the summation of the infinite series of the integer powers of the reciprocals of the positive integers. That is, the evaluation of

$$S_p = \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ for } p = 1, 2, 3, \ldots.$$

The answer for $p = 1$, which results in the so-called *harmonic series*, has been known since about 1350 to *diverge*, a result first shown by the medieval French mathematician and philosopher Nicole Oresme (1320–82).

This conclusion for S_1 surprises most people when they first encounter it, but Oresme's proof of it is beautifully simple. One simply writes S_1 as

$$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

and then replaces each term in each grouping on the right with the last (smallest) term in that grouping; notice that this last term will always be of the form $1/2^m$ where m is some integer. This process gives a lower bound on S_1, and so we have

$$S_1 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.$$

That is, we can add $\frac{1}{2}$ to the lower bound on S_1 as many times as we wish, which is just another way of saying that the lower bound itself diverges. But then S_1 must diverge, too.

The divergence is incredibly slow, however. For the partial sum of S_1 to exceed 15, for example, requires well over 1.6 million terms; after 10 *billion*

terms the partial sum is only about 23.6, and to reach 100 requires over 1.5×10^{43} terms. Finally, because of its connection to Euler, I should tell you that in 1731 he found that, if $S_1^{(n)}$ is the nth partial sum of S_1, then $\lim_{n\to\infty} \{S_1^{(n)} - \ln(n)\}$ *does* converge, to a number γ now called *Euler's constant*, which is $\gamma = 0.577215664901532\ldots$. After π and e, γ is perhaps the most important mathematical constant not appearing in elementary arithmetic. In 1735 Euler calculated γ to the fifteen correct decimal places given above, while in modern times it has been calculated to many thousands of places.

There is an elegant way to express γ in terms of S_p, using the power series expansion for $\ln(1 + z)$. This expansion is easily derived for all real z such that $-1 < z < 1$, just as the Danish mathematician Nicolaus Mercator (1619–87) did it in his 1668 book *Logarithmotechnia*. Write $1/(1 + z) = 1 - z + z^2 - z^3 + z^4 - \cdots$, which you can verify by long division. Then integrating both sides gives

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 \cdots + K,$$

where K is the indefinite constant of integration. But since at $z = 0$ we have $\ln(1) = 0$, we must then have $K = 0$, and we are done.

If you now successively substitute the values of $z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ into the above expression, you can write the following formulas:

$$1 = \ln(2) + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots,$$

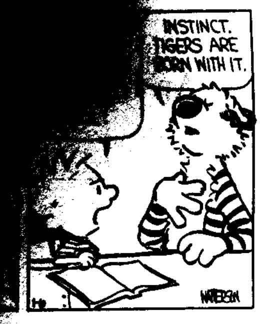
$$\frac{1}{2} = \ln\left(\frac{3}{2}\right) + \frac{1}{2}\cdot\frac{1}{2^2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{4}\cdot\frac{1}{2^4} - \frac{1}{5}\cdot\frac{1}{2^5} + \cdots,$$

$$\frac{1}{3} = \ln\left(\frac{4}{3}\right) + \frac{1}{2}\cdot\frac{1}{3^2} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{4}\cdot\frac{1}{3^4} - \frac{1}{5}\cdot\frac{1}{3^5} + \cdots,$$

$$\frac{1}{n} = \ln\left(\frac{n+1}{n}\right) + \frac{1}{2}\cdot\frac{1}{n^2} - \frac{1}{3}\cdot\frac{1}{n^3} + \frac{1}{4}\cdot\frac{1}{n^4} - \frac{1}{5}\cdot\frac{1}{n^5} + \cdots,$$

If you add these relations together then all the logarithmic terms cancel except one (the sum is said to *telescope*), and you will get

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln(n+1) = \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right)$$
$$- \frac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3}\right)$$
$$+ \frac{1}{4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{n^4}\right) - \cdots.$$



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